Exam. Code : 103206
Subject Code : 1232

B.A./B.Sc. 6th Semester

MATHEMATICS

Paper—I (Linear Algebra)

Time Allowed—Three Hours] [Maximum Marks—50

Note :— Attempt **FIVE** questions in all selecting at least **TWO** questions from each section.

SECTION—A

1. (a) Prove that Q^+, the set of positive rational numbers is an abelian group w.r.t. the binary operation $*$ defined as $a * b = \frac{ab}{3} \ \forall\ a, b \in Q^+$.

   (b) Prove that the necessary and sufficient condition for non-empty subset W of a vector space V(F) to be a subspace of V is that $\alpha x + \beta y \in W$ for all $\alpha, \beta \in F$ and $x, y \in W$.

2. (a) In the vector space $\mathbb{R}^3$ let $x = (1, 2, 1)$; $y = (3, 1, 5)$; $z = (3, -4, 7)$, prove that the subspaces spanned by $S = \{x, y\}$ and $T = \{x, y, z\}$ are same.

(b) If V(F) be a vector space then prove that the set S of non-zero vectors $v_1, v_2, \ldots, v_n \in V$ is linearly independent iff some vector $v_k \in S$, $2 \le k \le n$, can be expressed as a linear combination of its preceding vectors.

3. (a) Show that the vectors $u = (1 + i, 2i)$ and $v = (1, 1 + i)$ in $V_2(c)$ are L.D. but in $V_2(\mathbb{R})$ are L.I.

(b) Prove that any two bases of a finite dimensional vector space have same number of elements.

4. (a) Let V be a vector space of 2×2 symmetric matrices over $\mathbb{R}$. Find a basis and dimension of V.

(b) If V and W are subspaces of a finite dimensional vector space V(F), prove that :

$$\dim(V + W) = \dim V + \dim W - \dim (V \cap W).$$

5. (a) Let W_1 and W_2 be two subspaces of R^4, where

$$W_1 = \{(a, b, c, d) : b - 2c + d = 0\}$$

and $W_2 = \{(a, b, c, d) : a = d, b = 2c\}$

Find a basis and dimension of (i) W_1, (ii) W_2, (iii) $W_1 \cap W_2$.

(b) If W is a subspace of finite dimensional vector space V(F), prove that $\dim V/W = \dim V - \dim W$.

SECTION—B

6. (a) Find a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that $T(1, 0) = (1, 1)$ and $T(0, 1) = (-1, 2)$. Prove that T maps the square with vertices (0, 0), (1, 0), (1, 1) and (0, 1) in a parallelogram.

   (b) Show that a mapping $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $T(x, y, z) = (x + 1, y + z)$, is not a linear transformation.

7. (a) If V(F) and W(F) are vector spaces over the same field F and $T : V \to W$ is a L.T., prove that :

   (i) Range of T is a subspace W(F)

   (ii) Null space of T is a subspace V(F).

   (b) Find a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ whose range space is generated by $(1, 0, -1)$ and $(1, 2, 2)$.

8. (a) Prove that every n-dimensional vector space over the field F is isomorphic to space F^n.

   (b) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $T(x, y, z) = (x - 3y - 2z, x - 4z, y)$. Prove that T is invertible and find T^{-1}.

9. (a) Let $B = \{v_1, v_2, \ldots, v_n\}$ be a basis for vector space V(F) and $T : V \to V$ be a linear transformation. Prove that for any $v \in V$, $[T; B][v; B] = [T(v); B]$.

(b) Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be a L.T. defined by $T(x, y, z) = (3x + 2y - 4z, x - 5y + 3z)$. Find a matrix of T w.r.t. ordered bases $B_1 = \{(1, 1, 1), (1, 1, 0), (1, 0, 0)\}$ for $\mathbb{R}^3$ and $B_2 = \{(1, 3), (2, 5)\}$ for $\mathbb{R}^2$.

10. (a) Find a linear transformation $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ determined by the matrix $M = \begin{bmatrix} 1 & -1 \\ -2 & 3 \\ 0 & 1 \end{bmatrix}$ w.r.t. ordered basis $B_1 = \{(1, 1), (0, 2)\}$ for $\mathbb{R}^2$ and $B_2 = \{(0, 1, 1), (1, 0, 1), (1, 1, 0)\}$ for $\mathbb{R}^3$.

(b) Let $B_1 = \{(1, 0), (0, 1)\}$ and $B_2 = \{(1, 1), (-1, 0)\}$ be two ordered bases for $\mathbb{R}^2$. Find transition matrix P from B_1 to B_2, transition matrix Q from B_2 to B_1. Verify that $Q = P^{-1}$ and show that :

$$P[v; B_2] = [v; B_1] \ \forall\ v \rightarrow \mathbb{R}^2.$$